Heisenberg's eddy-viscosity approximation, the distant-interaction algorithm, and the ϵ expansion in turbulence

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Within the Heisenberg eddy-viscosity approximation, we apply Kraichnan's distant-interaction algorithm together with the ϵ expansion of Yakhot and Orszag [J. Sci. Comput. **1** (1), 3 (1986)]. This yields, in the leading order, a value of the Kolmogorov constant in exact agreement with that of Yakhot and Orszag's renormalization-group calculations. Various important features regarding the ϵ expansion are brought out through the stages of the approximations involved. The ϵ expansion is found to be an expansion in powers of $\epsilon/3$ rather than simply ϵ. A jump discontinuity in the spectrum is seen to be smoothed out by the ϵ expansion. Further, the extrapolation parameter of the distant-interaction algorithm becomes irrevelant in the leading order of the ϵ expansion.

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I. INTRODUCTION

The first phenomenological approach to the problem of homogeneous isotropic strong turbulence in fluids was offered by Kolmogorov and Obukhov [1,2]. The underlying mechanism of the energy cascade from large to small scales was hypothesized to be *local* in nature, leading to the energy spectrum

$$E(k)=C\overline{\varepsilon}^{2/3}k^{-5/3} \tag{1}$$

(in the universal inertial range) mainly based on dimensional grounds. Here $\overline{\varepsilon}$ is the mean energy injection rate to the fluid at large scales, k the inverse length scale, and C the universal Kolmogorov constant.

Attempts have been made to calculate the universal number C from a stochastic version of the Navier-Stokes dynamics

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\frac{\nabla P}{\rho}+\nu_0\nabla^2\mathbf{u}+\mathbf{f} \tag{2}$$

[$\mathbf{u}(\mathbf{x},t)$ and $P(\mathbf{x},t)$ being the velocity and pressure fields, ρ the density; accompanied by the incompressibility condition $\nabla\cdot\mathbf{u}=0$] where the fluid is assumed to be driven by a random external stirring force $\mathbf{f}(\mathbf{x},t)$. In particular, Yakhot and Orszag [3] took up the dynamic renormalization-group (RG) approach developed by Ma and Mazenko [4] and Forster, Nelson, and Stephen [5] and used it in the randomly stirred model of DeDominicis and Martin [6] where the correlation of the external stirring force is assumed to have a Gaussian white-noise statistics with the correlation

$$\langle f_i(\mathbf{k},\omega)f_j(\mathbf{k}',\omega')\rangle = F(k)P_{ij}(\mathbf{k})[2\pi]^d\delta^d(\mathbf{k}+\mathbf{k}')[2\pi]\delta(\omega+\omega')$$

in the Fourier space, with

$$F(k)=\frac{2D_0}{k^{d-4+\epsilon}}, \tag{3}$$

the space dimension being d. The approximations in the calculation involved nonlocal (distant) interactions among the Fourier modes and a procedure of ϵ expansion. This yielded a numerical value of C in remarkable agreement with the experimental values ($C=1.44,1.74$) [7,8] when ϵ was set equal to 4 in the leading contributing order.

Kraichnan [9,10], in order to analyze the effect of such nonlocal interactions, formulated the distant-interaction algorithm (DSTA), which involved approximations similar to Yakhot and Orszag's calculations. He applied the DSTA to a non-RG closure based model [explicitly, the energy equation of the direct interaction approximation (DIA)] and obtained an asymptotic expression for the eddy viscosity $\nu(k|p')$ in the Heisenberg approximation [11]. The ensuing results were *different*, although not far, from that of Yakhot and Orszag. It may be noted that Kraichnan's analysis of the Heisenberg eddy viscosity under the DSTA did not involve any ϵ expansion. Rather, ϵ was fixed at 4 (effectively) right from the beginning. However, it would be interesting to see the effect of ϵ expansion together with the approximations of the DSTA [12].

In this paper we use the DSTA in the energy equation of the DIA to get an asymptotic expression of the Heisenberg eddy viscosity for space dimension d. Remaining within this eddy-viscosity picture, we carry out the ϵ expansion of Yakhot and Orszag and calculate an amplitude ratio in the leading order. The energy integral being UV marginal at $\epsilon=0$, the ϵ expansion can be interpreted as a Laurent expansion about the UV pole, with $O(1/\epsilon)$ in the leading order. It is observed that this expansion is basically in powers of $\epsilon/3$ rather than just ϵ. This yields, in the leading order and for $\epsilon=4$, a value of C in exact agreement with that of Yakhot and Orszag. This agreement is important beacause we have based our calculation on the energy equation of the DIA, which does not involve the band-elimination technique of the RG.

Further, we observe various important features of the ϵ expansion displayed through the stages of the approxima-

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tions. The jump discontinuity in the energy spectrum is found to be smoothed out in the leading order (coming from the UV pole) of the ϵ expansion. Importantly, it is explicitly shown that the DSTA extrapolation parameter λ (Kraichnan's β), which may be assumed to measure the strength of nonlocality in the approximation, becomes irrelevant in the leading order of the ϵ expansion.

II. DISTANT-INTERACTION ALGORITHM

For the triad interaction $\mathbf{p}+\mathbf{q}=\mathbf{k}$, with $\mathbf{k}$ being the external wave vector, the DSTA [9,10] can be summarized as follows.

(1) Consider the eddy viscosity felt at the wave number k due only to wave numbers $(p,q)>p'$, denoted by

$$\nu(k|p') \quad \text{for} \quad p' \gg k.$$

Thus the effect of $(p,q)<p'$ is entirely ignored.

(2) Extrapolate p' to a wave number λk, with $\lambda \geqslant 1$ an $O(1)$ numerical constant, to get

$$\nu(k|\lambda k) = \lim_{p' \to \lambda k} \nu(k|p').$$

(3) Identify the eddy viscosity at k as

$$\nu(k) \equiv \nu(k|\lambda k).$$

Thus in the DSTA, the calculations are performed in a region of highly nonlocal interactions, and then the results are extrapolated to a region of local interactions. We shall call λ the extrapolation parameter of the DSTA. It may be assumed that the numerical value of λ measures the strength of nonlocality in the approximation under the DSTA.

Kraichnan basically used the idea of eddy viscosity obtained by applying the DSTA to the energy-transfer equation rather than the response equation of the direct interaction approximation [13], thereby avoiding the added problem of infrared divergence for $\epsilon \geqslant 3$ (discovered by Edwards [14]) with the response integral of the DIA, while the Kolmogorov scaling [Eq. 1] occurs for $\epsilon = 4$.

III. CONSERVED ENERGY TRANSFER

The DIA energy-transfer equation [13,15] can be written as

$$\left(\frac{\partial}{\partial t} + 2\nu_0 k^2 \right) E(k;t,t) = T(k;t,t) \tag{4}$$

where the transfer $T(k)$ (suppressing the index t), in d dimensions, is given by [16]

$$T(k) = \frac{8k^2}{(d-1)^2} k^{d-1} \int \frac{d^d p}{S_d} \theta(k,p,q) \times \left[a(k,p,q) \frac{E(p)}{p^{d-1}} \frac{E(q)}{q^{d-1}} - \frac{1}{2} b(k,q,p) \frac{E(p)}{p^{d-1}} \frac{E(k)}{k^{d-1}} - \frac{1}{2} b(k,p,q) \frac{E(q)}{q^{d-1}} \frac{E(k)}{k^{d-1}} \right], \tag{5}$$

assuming $\theta(k,p,q)$ to be symmetric in all three indices. For exponential decay of the response and correlation

$$\theta(k,p,q) = \frac{1}{\nu(k)k^2 + \nu(p)p^2 + \nu(q)q^2} \tag{6}$$

in the Edwards' simplified assumption of generalized fluctuation-dissipation theorem (FDT) [14]. The geometrical coefficients are given by

$$2k^2 b(k,q,p) = P_{ijl}(\mathbf{k}) P_{jm}(\mathbf{p}) P_{ln}(\mathbf{q}) P_{nmi}(\mathbf{q}), \tag{7}$$

$$2k^2 b(k,p,q) = P_{ijl}(\mathbf{k}) P_{jm}(\mathbf{q}) P_{ln}(\mathbf{p}) P_{nmi}(\mathbf{p}), \tag{8}$$

$$2a(k,p,q) = b(k,p,q) + b(k,q,p). \tag{9}$$

After some algebra, it can be shown that [16]

$$b(k,q,p) = \frac{q}{k}(y^3 + xz) + \frac{d-3}{2}(1-z^2) \tag{10}$$

and

$$b(k,p,q) = \frac{p}{k}(z^3 + xy) + \frac{d-3}{2}(1-y^2) \tag{11}$$

where

$$x = -\frac{\mathbf{p}\cdot\mathbf{q}}{pq}, \quad y = \frac{\mathbf{k}\cdot\mathbf{q}}{kq}, \quad z = \frac{\mathbf{k}\cdot\mathbf{p}}{kp}. \tag{12}$$

Net energy transfer rate across a wave number j is defined as [13]

$$\Pi(j) = \int_j^\infty T_{(p,q)<j}(k)\,dk - \int_0^j T_{(p,q)>j}(k)\,dk, \tag{13}$$

where the inequalities expressed as subscripts to $T(k)$ refer to the region of the (p,q) integration in Eq. (5), with the triangle condition $\mathbf{p}+\mathbf{q}=\mathbf{k}$.

Noting that $E(k) \sim k^{d-1} \int_{-\infty}^{+\infty} (d\omega/[2\pi]) |G(k,\omega)|^2 F(k)$ with the renormalized propagator $G(k,\omega) = 1/[-i\omega + k^2 \nu(k)]$, we get $E(k) \sim k^{d-1} F(k)/k^2 \nu(k)$. Using Eq. (3) this yields

$$E(k) \sim \frac{k^{1-\epsilon}}{\nu(k)}. \tag{14}$$

On the other hand, when scaling arguments are applied to Eq. (22), we get

$$\nu(k) \sim \frac{E(k)}{k\nu(k)}. \tag{15}$$

The above two equations yield the following scaling relations:

$$E(k) \sim k^{1-2\epsilon/3}, \qquad \nu(k) \sim k^{-\epsilon/3}. \tag{16}$$

Thus the naive dimension of the above flux integral, Eq. (13), is found to be

$$\Pi(j)\sim j^{4-\epsilon}, \tag{17}$$

when Eqs. (5), (6), and (16) are used in Eq. (13). We must have $\Pi(j)=\bar{\varepsilon}$ because $\bar{\varepsilon}$ is the constant rate of energy transfer in the inertial range. Thus conserved transfer of energy demands that $\Pi(j)$ be wave number independent, and hence clearly demands $\epsilon=4$ in the inertial range.

When we use the scaling relations from Eq. (16) in the transfer integral in Eq. (5), we find that it is both UV and IR convergent in the range $0<\epsilon<6$. Therefore this integral does not pose any problem for the Kolmogorov value $\epsilon=4$, which asserts conserved transfer of energy, as seen from Eq. (17).

IV. EDDY VISCOSITY

In this section we shall derive an asymptotic form of the eddy viscosity from the energy-transfer equation in d dimensions, and analyze some of the effects of ϵ expansion. Here the $p\gg k$ expansions are different from Kraichnan's expansions in the sense that they exploit the d-dimensional angular integrals directly. We shall try to make the presentation of these expansions somewhat in detail in the following in order to bring out some of the features of the ϵ expansion to be carried out in the next section.

Kraichnan [17] defined the eddy viscosity $\nu(k|p')$ felt at k due to interactions only with $(p,q)>p'$ in the Heisenberg approximation [11]:

$$T(k|p')=-2\nu(k|p')k^2E(k), \tag{18}$$

which is obtained from the transfer integral $T(k)$ by setting an IR cutoff at p'. Then from Eq. (13) the energy-transfer rate to all wave numbers $(p,q)>p'$ from distant wave numbers $k<k'$ is

$$\Pi(k'|p')=-\int_0^{k'}T(k|p')dk;p'\gg k', \tag{19}$$

the first integral in Eq. (13) being negligible because it involves the region $k>p'$ and $(p,q)<k'$, where the triangle condition can hardly be satisfied in the limit $p'\gg k'$.

We now turn to calculate $T(k|p')$ from $T(k)$ with a lower cutoff at p' and invoking the DSTA. In the limit $p'\gg k$ we have $p\approx q$, and hence

$$\frac{E(p)}{p^{d-1}}\approx\frac{E(q)}{q^{d-1}}\ll\frac{E(k)}{k^{d-1}}\ \text{ for any }\ d>2-2\epsilon/3. \tag{20}$$

Thus only the terms containing $E(k)$ in Eq. (5) dominate, leading to

$$T(k|p')=-\frac{4k^2E(k)}{(d-1)^2}\int_{p'}^{\infty}dp\oint\frac{d\Omega}{S_d}\theta(k,p,q)\times\left[b(k,q,p)E(p)+b(k,p,q)\frac{E(q)}{q^{d-1}}p^{d-1}\right], \tag{21}$$

which, upon using the Heisenberg approximation, Eq. (18), yields

$$\nu(k|p')=\frac{2}{(d-1)^2}\int_{p'}^{\infty}dp\oint\frac{d\Omega}{S_d}\theta(k,p,q)\left[b(k,q,p)E(p)+b(k,p,q)\frac{E(q)}{q^{d-1}}p^{d-1}\right]. \tag{22}$$

This integral is to be expanded in the limit $p\gg k$. The Taylor expansions for the quantities in the integral can be calculated from Eqs. (10)–(12), (6), and (16) to be

$$b(k,q,p)=\frac{p}{k}z(1-z^2)+2z^2(1-z^2)+\left(\frac{d-3}{2}\right)(1-z^2)+\cdots, \tag{23}$$

$$b(k,p,q)=-\frac{p}{k}z(1-z^2)+\left(\frac{d-1}{2}\right)(1-z^2)+\cdots, \tag{24}$$

$$\theta(k,p,q)=\theta(0,p,p)\left\{1+\left(1-\frac{\epsilon}{6}\right)\frac{k}{p}z+\cdots\right\}, \tag{25}$$

$$E(q)=E(p)-kz\frac{\partial E(p)}{\partial p}+\cdots, \tag{26}$$

$$\frac{1}{q^{d-1}}=\frac{1}{p^{d-1}}\left\{1+(d-1)\frac{k}{p}z+\cdots\right\}, \tag{27}$$

where $z=\cos\theta$, θ being the angle between **k** and **p**. Combining them we get

$$\theta(k,p,q)b(k,q,p)=\theta(0,p,p)\left\{\frac{p}{k}z(1-z^2)+\left(3-\frac{\epsilon}{6}\right)z^2(1-z^2)+\left(\frac{d-3}{2}\right)(1-z^2)+\cdots\right\}, \tag{28}$$

$$\theta(k,p,q)b(k,p,q)\frac{E(q)}{q^{d-1}}=\frac{\theta(0,p,p)}{p^{d-1}}\left[\left\{-\frac{p}{k}z(1-z^2)+\left(\frac{d-1}{2}\right)(1-z^2)-\left(d-\frac{\epsilon}{6}\right)z^2(1-z^2)\right\}E(p)+z^2(1-z^2)\,p\frac{\partial E(p)}{\partial p}+\cdots\right]. \tag{29}$$

Using the expansions from Eqs. (28) and (29) we find

$$\theta(k,p,q)\left[b(k,q,p)E(p)+b(k,p,q)\frac{E(q)}{q^{d-1}}p^{d-1}\right]$$
$$=\theta(0,p,p)\left[\{(d-2)(1-z^2)+(3-d)z^2(1-z^2)\}E(p)+z^2(1-z^2)p\frac{\partial E(p)}{\partial p}+\cdots\right], \quad (30)$$

whence we make the critical observation that the $O(\epsilon)$ contributions to the first and second terms of the left-hand side *cancel* each other. Therefore ϵ expansion (setting $\epsilon=0$ in the coefficients) at this stage does not make any difference. The only $O(\epsilon)$ coefficient that can appear is from the term

$$\frac{\partial E(p)}{\partial p}=\left(1-\frac{2\epsilon}{3}\right)\frac{E(p)}{p}+\delta(p-p')E(p). \quad (31)$$

where the term involving $\delta(p-p')$ appears because the spectrum jumps from zero to the Kolmogorov value at $p=p'$ in the eddy-viscosity picture [9,10]. It may be noted that the $O(\epsilon^2)$ and higher order terms occur with powers of k/p, higher than the leading contributing order, $O(k^0/p^0)$.

Making use of the formulas for angular integration,

$$\oint d\Omega=S_d,\quad \oint z^2 d\Omega=S_d/d,\quad \oint z^4 \mathrm{d}\Omega=3S_d/d(d+2), \quad (32)$$

where $S_d=2\pi^{d/2}/\Gamma(d/2)$, the surface area of a unit sphere embedded in a d-dimesional space, the angular integration of Eq. (30) yields

$$\oint\frac{d\Omega}{S_d}\ [\text{Eq. (30)}]=\theta(0,p,p)\frac{d-1}{d(d+2)}\left[(d^2-d-1)E(p)+p\frac{\partial E(p)}{\partial p}\right], \quad (33)$$

whence Eq. (22) yields

$$\nu(k|p')=\left(\frac{2}{d-1}\right)\frac{1}{d(d+2)}\int_{p'}^{\infty}dp\ \theta(0,p,p)\times\left[(d^2-d-1)E(p)+p\frac{\partial E(p)}{\partial p}\right]. \quad (34)$$

We see that the factor (d^2-d-1) is a consequence of *exact cancellation* of the $O(\epsilon)$ terms. Thus this factor is the same with or without ϵ expansion.

This result matches those of Kraichnan for both $d=3$ and 2. For comparison see Eq. (3.5) for $d=3$ and Eq. (4.6) for $d=2$ in Ref. [17]. The latter result can easily be cast into the present form after some algebra.

Now as $E(p)=\Lambda_0 p^{1-2\epsilon/3}$, using Eq. (31), Eq. (34) reduces to

$$\nu(k|p')=\frac{2}{d-1}\left[A_d(\epsilon)\int_{p'}^{\infty}dp\ \theta(0,p,p)E(p)+B_d p'\theta(0,p',p')E(p')\right] \quad (35)$$

where

$$A_d(\epsilon)=\frac{d^2-d-2\epsilon/3}{d(d+2)};\quad B_d=\frac{1}{d(d+2)}. \quad (36)$$

It may be noted that the $O(\epsilon)$ term here is different from that in the renormalization-group calculations [3,18].

Power counting in this integral shows that it is UV convergent for $\epsilon>0$. Finally, carrying out the integration in Eq. (35), which gives a factor $O(1/\epsilon)$, we obtain

$$\nu(k|p')=\frac{2}{d-1}\left[\frac{3}{\epsilon}A_d(\epsilon)+B_d\right]p'\theta(0,p',p')E(p'). \quad (37)$$

V. ϵ EXPANSION

Noting that the integrals in Eqs. (5), (22), and (35) are UV marginal at $\epsilon=0$, we now carry out the ϵ expansion about $\epsilon=0$ with this expression for eddy viscosity, Eq. (37). Using step (3) of the DSTA, and using $\theta(0,p',p')=1/2\nu(p')p'^2$, we get

$$\nu(k)\equiv\nu(k|\lambda k)=\frac{2}{d-1}\left[\frac{3}{\epsilon}A_d(\epsilon)+B_d\right]\frac{E(\lambda k)}{2\lambda k\nu(\lambda k)}. \quad (38)$$

Using the scaling relations

$$E(q)=C\overline{\varepsilon^{2/3}}q^{1-2\epsilon/3},\qquad \nu(q)=\alpha\overline{\varepsilon^{1/3}}q^{-\epsilon/3}, \quad (39)$$

as suggested by Eq. (16), Eq. (38) leads to

$$\frac{\alpha^2}{C}=\frac{1}{d-1}\left[\frac{3}{\epsilon}A_d(\epsilon)+B_d\right]\lambda^{-\epsilon/3}. \quad (40)$$

Now we note that λ is an irrelevant parameter in the sense that under ϵ expansion

$$\lambda^{-\epsilon/3}=1+O(\epsilon), \quad (41)$$

so that

$$A_d(\epsilon)\lambda^{-\epsilon/3}=A_d(0)+O(\epsilon), \quad (42)$$

leading to

$$\frac{\alpha^2}{C}=\frac{1}{d-1}\left[\frac{3}{\epsilon}A_d(0)+O(\epsilon^0)\right]. \quad (43)$$

As the integrals in Eqs. (5), (22), and (35) are UV marginal at $\epsilon=0$, this is a Laurent expansion in ϵ about the UV pole at $\epsilon=0$, the leading order being $O(1/\epsilon)$.

We observe from Eqs. (40), (41), (36), and (42) that ϵ occurs with a factor of 1/3 in all places. Therefore the ϵ expansion in Eq. (43) is actually an expansion in powers of

$\epsilon/3$ rather than just ϵ. This suggests that the range of validity of the ϵ expansion can be trusted over a much wider range of values of ϵ than just small values close to zero. It may further be noted that the normal ϵ expansion in a RG analysis based on a shell-elimination process from the viscous end yields an ϵ expansion in powers of ϵ rather than $\epsilon/3$, requiring small values of ϵ for its validity. The present ϵ expansion therefore offers a better confidence in the ϵ expansion procedure.

Noting from Eq. (36) that $A_3(0)=2/5$, and Eq. (43) requiring no particular value of λ in the leading order, we therefore get

$$\frac{\alpha^2}{C}=\tfrac{1}{2}\tfrac{3}{4}A_3(0)=\tfrac{3}{20} \tag{44}$$

for $d=3$, Kolmogorov spectrum, $\epsilon=4$, in the leading order in the ϵ expansion. It may be noted that the jump discontinuity associated with the energy spectrum in the eddy-viscosity picture (the term containing B_d) does not contribute in the leading order of the ϵ expansion.

When Eq. (44) is coupled with Kraichnan's [19] numerical result (which is also used by Yakhot and Orszag in their calculations [3] as an input)

$$\frac{\alpha}{C^2}=\frac{1}{(3.022)^{3/2}}=0.1904 \tag{45}$$

(see Leslie [15]), one finally gets

$$C=\frac{(\alpha^2/C)^{1/3}}{(\alpha/C^2)^{2/3}}=\frac{(3/20)^{1/3}}{(0.1904)^{2/3}}=3.022\left(\frac{3}{20}\right)^{1/3}=1.6057. \tag{46}$$

Upon carrying out some algebra it can be seen that the final results of Yakhot and Orszag's RG theory produce the second equality $C=(3/20)^{1/3}/(0.1904)^{2/3}$ *exactly* (at $d=3$), whereas the last equality $C=1.6057$ does not match with their final result, $C=1.617$; there seems to have been a numerical error in the latter.

The agreement with the RG result of Yakhot and Orszag is because of the fact that the leading order contribution in the ϵ expansion remains the same in the present calculation, together with the fact that we have used the same numerical result of Eq. (45) as has been used by Yakhot and Orszag.

At this point it may be noted that the DSTA extrapolation parameter λ can be set equal to unity along with the choice of making no ϵ expansion, and setting $\epsilon=4$, in order to cross-check our result with that of Ref. [9]. This would yield

$$\frac{\alpha^2}{C}=\tfrac{1}{2}[\tfrac{3}{4}A_3(4)+B_3]\lambda^{-4/3}. \tag{47}$$

Noting that $A_3(4)=10/45$, $B_3=1/15$, and $\alpha^2/C=(7/60)\lambda^{-4/3}$ (in agreement with Kraichnan's result [9]), and setting $\lambda=1$ yields $C=1.4764$. This result turns out to be different from that in Ref. [9], namely, $C=1.5618$, under the same substitutions.

The difference arises because Kraichnan does not use the exact numerical value from Eq. (45) in the derivation in Ref. [9]. Effectively, an approximate value for α/C^2 was calculated analytically by making approximations to the flux integral $\Pi(j)$. On the other hand, we have used the numerical value from Eq. (45), which leads to a different result.

VI. CONCLUSIONS

We have used the energy equation of the DIA [expressed in a general d space dimensions, Eq. (4)] and assumed that Heisenberg's eddy-viscosity assumption, Eq. (18), holds. The transfer integral, Eq. (5), being both UV and IR convergent in the region $0<\epsilon<6$, the calculations are safe for the Kolmogorov value $\epsilon=4$, dictated by the condition of conserved energy transfer Eq. (17). Remaining within such a closure scheme, we have applied Krichnan's DSTA and obtained an asymptotic expression of the Heisenberg eddy viscosity in space dimension d, which led to an expression for the amplitude ratio α^2/C, Eq. (40). Performing Yakhot and Orszag's ϵ expansion about $\epsilon=0$ now leads to Eq. (43). Noting that the transfer intergal is UV marginal at $\epsilon=0$, this expansion is basically a Laurent expansion about the UV pole at $\epsilon=0$, the leading order being $\sim 1/\epsilon$. Substituting the Kolmogorov value $\epsilon=4$ in the leading order, we have combined the result with another exactly known amplitude ratio α/C^2 [Eq. (45)]. This yielded a value of the Kolmogorov constant C [Eq. (46)] in exact agreement with Yakhot and Orszag's RG calculations, although the calculations did not involve the band-elimination technique of the RG.

The agreement may be attributed to the following facts: (1) the DSTA embodies similar nonlocal approximations as in Yakhot and Orszag's calculations; (2) the ϵ expansion procedure smooths out the jump discontinuity in the energy spectrum in the eddy-viscosity picture in the sense that the corresponding contribution is pushed into the subleading order with respect to the UV pole of the ϵ expansion; (3) we have made use of Eq. (45).

Without making the ϵ expansion and fixing $\epsilon=4$, and setting the extrapolation parameter $\lambda=1$, yielded the value $C=1.4764$. The disagreement with Kraichnan's result $C=1.5618$ (with similar substitutions) can be attributed to the fact that Kraichnan does not use Eq. (45) in his calculations. This further suggests that an ϵ expansion in the framework of Refs. [9,10], which does not use Eq. (45), would not reproduce Yakhot and Orszag's value for C.

We have also made various important points about the ϵ expansion through the stages of the approximations. It is seen after the Taylor expansion in Eq. (22) that two $O(\epsilon)$ terms exactly cancel out, and only the term $\partial E/\partial p$ gives a term of $O(\epsilon)$. Further it was noted that the $O(\epsilon^2)$ and higher order terms occur with powers of k/p, higher than the leading contributing order, $O(k^0/p^0)$, of the DSTA.

It was further observed after Eq. (36) that the $O(\epsilon)$ term of $A_d(\epsilon)$ is different from that in the renormalization-group calculations. The difference may be attributed to the fact that we have used a closure model in the calculations.

An important result is that it is explicitly seen in Eq. (43) that the DSTA extrapolation parameter λ (Kraichnan's β) becomes irrelevant under the ϵ expansion, so that the numerical value of λ becomes irrelevant in the leading order.

We have also observed that the ϵ expansion is basically an expansion in powers of $\epsilon/3$ rather than just ϵ. Whether it has any possible connection with the IR divergences for ϵ

$\geqslant 3$ associated with the response integral of the DIA remains an open query. We could, however, bypass the danger of encountering the IR divergence by using the energy equation of the DIA, which is devoid of any such divergences.

Finally, we note that the intermittency corrections to the Kolmogorov scaling are neglected in our calculations, the underlying assumption being the validity of a closure approximation. Relevant calculations corresponding to such corrections would presumably require a formulation beyond the presently available closure schemes where the delicate nonlinear interactions in the dynamics giving rise to such corrections are not suppressed by the analytical averaging procedure.

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